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Power-Law Falloff in Two-Dimensional Coulomb Gases at Inverse Temperature $\beta > 8\pi$

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We consider two-dimensional Coulomb gases (standard or hard-core) with activity z, and prove that for any $\beta > 8\pi$ the two-point external charges correlation function exhibits the power-law falloff characteristic of the Kosterlitz–Thouless phase at sufficiently low activity.

KEY WORDS: Coulomb gas; Kosterlitz–Thouless phase; critical temperature; multiscale analysis.

1. INTRODUCTION

A two-dimensional (lattice) Coulomb gas is a system of classical particles with electric charges ± 1, whose possible positions range over a finite array of sites $\Lambda \subset \mathbf{Z}^2$, interacting via a two-body Coulomb potential. In the sine-Gordon representation the partition function of the gas is given by

$$Z_\Lambda = \int \prod_{x \in \Lambda} \hat{\lambda}_z(\phi(x))\, d\mu_\beta(\phi) \tag{1.1}$$

where $d\mu_\beta$ is the Gaussian measure with covariance $\beta(-\Delta)^{-1}$, β is the inverse temperature, Δ is the finite-difference Laplacian on $\mathbf{Z}^2$, and

$$\hat{\lambda}_z(\phi) = \sum_{q \in \mathbf{Z}} \lambda_z(q) e^{iq\phi}$$

where λ_z is the "*a priori*" charge weight at particle activity $z \geqslant 0$.

In this article we will always require λ_z to satisfy:

(a) $\lambda_z(q) = \lambda_z(-q)$.

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(b) $|\lambda_z(q)| \leqslant c(z)e^{-\nu|q|}$ for all $q \in \mathbf{Z}\backslash\{0\}$, where $\nu > 0$ and $\lim_{z \downarrow 0} c(z) = 0$.

Such λ_z will be called *standard*. The typical examples are:

1. The hard-core gas:

$$\lambda_z(q) = \begin{cases} 1, & q = 0 \\ z/2, & q = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

In this case, $\hat{\lambda}_z(\phi) = 1 + z \cos \phi$.

2. The standard gas:

$$\lambda_z(q) = \frac{1}{2\pi} \int_0^{2\pi} e^{z \cos \theta} \cos q\theta \, d\theta$$

In this case, $\hat{\lambda}_z(\phi) = e^{z \cos \phi}$.

The external charges correlation function is defined by

$$G_{\xi, \Lambda}(x) = \frac{Z_{\xi, \Lambda}(x)}{Z_\Lambda}$$

where

$$Z_{\xi, \Lambda}(x) = \int e^{i\xi(\phi(0) - \phi(x))} \prod_{y \in \Lambda} \hat{\lambda}_z(\phi(y)) \, d\mu_\beta(\phi) \tag{1.2}$$

is the external charges partition function. Here $\xi \in \mathbf{R}$. By $G_\xi(x)$ we will denote a thermodynamic limit.

At high temperature, Brydges and Federbush[1] showed that Debye screening occurs (see also Yang[2]), i.e.,

$$G_\xi(x) \to Cte > 0 \qquad \text{as} \quad x \to \infty$$

exponentially fast.

An application of Jensen's inequality in the q-variables shows that[12]

$$G_\xi(x) \geqslant \frac{C_{\beta, \xi}}{|x|^{\beta\xi^2/2\pi}}$$

where $0 < C_{\beta, \xi} < \infty$.

Fröhlich and Spencer[3] established the existence of a Kosterlitz–Thouless transition from a high-temperature to a low-temperature phase, characterized by scaling and power falloff of correlations. They proved that Debye screening does not occur for β sufficiently large and

$$G_\xi(x) \leqslant \frac{C}{|x|^{c\beta\eta^2 - b}}$$

for $\beta > b/c\eta^2$, where $\eta = \min(\xi, (1-\xi))$ and $0 < \xi < 1$.

In a recent article in collaboration with Perez,[4] we improved on the Fröhlich–Spencer result by showing the existence of a critical inverse temperature $\bar{\beta} = \bar{\beta}(\lambda_z) < \infty$, and $\theta = \theta(\lambda_z) > 0$, such that for all $\beta > \bar{\beta}$ and $\xi \in \mathbf{R}$ we have

$$G_\xi(x) \leqslant \frac{C}{|x|^{\theta\beta\eta^2}}$$

where $\eta = \operatorname{dist}(\xi, \mathbf{Z})$, $\bar{\eta} = \operatorname{dist}(\xi, \mathbf{Z}\backslash\{0\})$, and $C = C(\beta, \bar{\eta})$ is such that

$$\sup_{\gamma < \bar{\eta} \leqslant 1} C(\beta, \bar{\eta}) < \infty \qquad \text{for any} \quad \gamma > 0$$

In addition, it was shown that for standard λ_z this critical inverse temperature $\bar{\beta} = \bar{\beta}(z)$ is at most 24π in the low-activity limit, i.e., $\bar{\beta}(0) \equiv \lim_{z \downarrow 0} \bar{\beta}(z) \leqslant 24\pi$. This result was improved to $\bar{\beta}(0) \leqslant 8(1+\sqrt{3})\pi/(3-\sqrt{3}) \approx 17.2\pi$ by Marchetti.[5]

A new proof of the Fröhlich and Spencer results was given recently by Braga.[6]

Renormalization group analysis and energy–entropy arguments suggest that $\bar{\beta}(0) = 8\pi$.[3,7–9] Recently Dimock and Hurd[10] proved that the eqilibrium measure of a standard Coulomb gas on $\mathbf{R}^2$ is driven, under a renormalization group transformation, to a Gaussian measure if $\beta > 8\pi$ and z is sufficiently small.

In this article we prove that $\bar{\beta}(0) \leqslant 8\pi$. More precisely, we have the following result.

Theorem 1.1. Consider a two-dimensional Coulomb gas with a standard "*a priori*" charge weight. Let $\beta > 8\pi$.

Then there exist $z(\beta) > 0$ and $\theta = \theta(\beta) > 0$ such that for $0 < z < z(\beta)$ we have

$$G_\xi(x) \leqslant \frac{C}{|x|^{\theta\eta^2}}$$

for all $\xi \in \mathbf{R}$, where $\eta = \operatorname{dist}(\xi, \mathbf{Z})$, $\bar{\eta} = \operatorname{dist}(\xi, \mathbf{Z}\backslash\{0\})$, and $C = C(\beta, z, \bar{\eta})$ with

$$\sup_{\gamma < \bar{\eta} \leqslant 1} C(\beta, z, \bar{\eta}) < \infty \qquad \text{for any} \quad \gamma > 0$$

To prove the theorem, we modify the procedure used in ref. 4. The procedure in ref. 4 used ideas developed for the hierarchical model by Marchetti and Perez[8,9] combined with the main ingredients of the Fröhlich-Spencer proof.[3] As in ref. 3, expectations in the two-dimensional Coulomb gas are written as convex combinations of expectations in diluted gases of neutral multipoles of variable sizes by a simple trigonometric identity. Falloff is extracted from charged multipoles by an analytic continuation argument. The partition function (with or without external charges) is initially rewritten as a convex combination of (appropriately defined) regular partition functions in a given initial scale. It is then proven that regular partition functions at a given scale can be written as convex combinations of regular partition functions at the next scale. The scales used are of the form $L_{k+1} \approx L_k^{\alpha}$, and in ref. 4 it was shown that

$$\bar{\beta}(0) \leqslant 8\pi \frac{\alpha}{\alpha - 2} \tag{1.3}$$

But the analytic continuation argument (Lemma 3.3 in ref. 4) required $\alpha > 3/2$. Hence $\bar{\beta}(0) \leqslant 24\pi$. The $\alpha > 3/2$ came from taking into account the background of neutral multipoles in the analytic continuation argument. By looking more closely at this background, Marchetti[5] showed that one needed only $\alpha > (1 + \sqrt{3})/2$ and hence $\bar{\beta}(0) \leqslant 8(1 + \sqrt{3})\pi/(3 - \sqrt{3}) \simeq 17.2\pi$.

It follows from (1.3) that to prove $\bar{\beta}(0) \leqslant 8\pi$, we need to let $\alpha \downarrow 1$. But we must pay a price for this. The proof of Lemma 3.3 in ref. 4 was based on an imaginary shift of the integration variable ϕ which was required to be constant on the support of the neutral multipoles in the background. The falloff factor obtained at scale L was of the form $Y \approx L^{-(\alpha-1)\beta/(4\pi + C)}$ for some constant $C = C(L, \alpha) \approx L^{3-2\alpha}$. To make $Y \to 0$ as $L \to \infty$, we needed $\alpha > 3/2$. It turns out that the inverse temperature $\bar{\beta}$ above which the proof in ref. 4 gives power-law falloff is

$$\bar{\beta} = (8\pi + 2C) \frac{\alpha}{2 - \alpha}$$

By taking the initial scale to infinity we get (1.3).

In this paper we improve the energy estimate to obtain

$$C = C(L, \alpha) \approx (\log L)^p \, L^{(1-\alpha)}$$

where $2 < p < \infty$ is fixed. We can thus pick any $\alpha > 1$. To do so we cannot make the imaginary shift constant on the neutral charges background, so we must consider a more general form of a regular partition function. This has been advocated by Spencer[11] (some preliminary calculations were done by Braga).

This paper is organized as follows. In Section 2 the partition function (1.2) with *standard "a priori"* charge weight is rewritten as a convex combination of *k-regular* partition functions for any scale k (Theorem 2.1). In Section 3 we prove the analogous result for the external charges partition function (Theorem 3.1) and then prove power decay for the external charges correlation function (Theorem 1.1). In Section 4 we prove Lemma 2.3, which gives the energy estimate that allows us to extract power-law falloff for any scale parameter $\alpha > 1$. We consider this improvement in the energy estimate the most important result in this work.

2. THE PARTITION FUNCTION

We will follow the notation used in ref. 4, except that we will use two norms in $\mathbf{Z}^2$: $|x|_2$ and $|x|_\infty$. By $B(x, L)$ we will denote the square in $\mathbf{Z}^2$ centered at x with side L, i.e.,

$$B(x, L) = \left\{ y \in \mathbf{Z}^2 : |y - x|_\infty < \frac{L}{2} \right\}$$

We will also use $\bar{B}(x, L) = B(x, \frac{4}{3}L)$.

We will fix $\beta > 8\pi$, and pick $\alpha > 1$ and an initial scale $L_1 = 3^{n_1}$, where $n_1 \in \mathbf{N}$. The successive scales are then given by $L_{k+1} = 3^{n_{k+1}}$, where $n_{k+1} = [\alpha n_k]$ (here $[t]$ = largest integer $\leqslant t$). We set $L_0 \equiv 1$.

We will always take Λ to be a square centered at 0, say $\Lambda = B(0, R)$, and we pick N such that $L_{N-1} < R \leqslant L_N$. We set $\Lambda^{(k)} = \Lambda \cap L_k \mathbf{Z}^2$, $B_k(y) = B(y, L_k)$ for $y \in \Lambda^{(k)}$, $B_k^{(k')}(y) = B_k(y) \cap L_{k'} \mathbf{Z}^2$ for $k' \leqslant k$. Notice $\Lambda^{(0)} = \Lambda$, $\Lambda^{(N)} = \{0\}$.

Let us fix a standard charge weight $\lambda_z(q)$; we set $\zeta(q) = c_1 e^{-(\nu/2)|q|}$, where c_1 is a constant chosen so $\sum_{q \in \mathbf{Z}} \zeta(q) = 1$. As in ref. 4, Section 2, we start by rewriting Z_Λ given by (1.1) as a convex combination of expressions of the form

$$\int \prod_{y \in \Lambda^{(1)}} (1 + z_y \cos \phi(\rho_y))\, d\mu_\beta(\phi) \tag{2.1}$$

where $\rho_y : \mathbf{Z}^2 \to \mathbf{Z}$ with supp $\rho_y \subset B_1(y)$, $\rho_y \not\equiv 0$, and

$$0 < z_y \leqslant \prod_{\substack{u \in B_1(y) \\ \rho_y(u) \neq 0}} \left[\frac{L_1^2}{\log 2} \frac{2\lambda(\rho_y(u))}{\zeta(\rho_y(u))} \right] \leqslant c_2 L_1^2 c(z) e^{-(\nu/2)|\rho_y|} \tag{2.2}$$

where $|\rho_y| \equiv |\rho_y|_1 = \sum_{u \in \mathbf{Z}^2} |\rho_y(u)|$, $c_2 = 2/(c_1 \log 2) < \infty$, in case $c_2 L_1^2 c(z) < 1$, which is always true for z sufficiently small.

We will need to perform imaginary shifts in ϕ in expressions like (2.1). This will change the form of the integrand in (2.1).

So given k, let $\varepsilon^{(k)}$ be the real-valued function on $\mathbf{Z}^2$ given by

$$\varepsilon^{(k)}(x)=\begin{cases}\log\dfrac{1}{15}\dfrac{L_{k+1}}{L_k} & \text{if } |x|_2\leqslant\dfrac{15}{6}L_k\\ \log\dfrac{L_{k+1}}{6|x|_2} & \text{if } \dfrac{15}{6}L_k\leqslant|x|_2\leqslant\dfrac{1}{6}L_{k+1}\\ 0 & \text{if } |x|_2\geqslant\dfrac{1}{6}L_{k+1}\end{cases} \tag{2.3}$$

For $y\in\mathbf{Z}^2$, $\varepsilon_y^{(k)}(x)=\varepsilon^{(k)}(x-y)$. We also define

$$\mathscr{F}_k=\{\varepsilon_y^{(l)};\, l>k,\ y\in\Lambda^{(l)}\}$$

Because the form of integrand in (2.1) will change, we need to generalize the notion of a weighed charge density (ρ, z) used in ref. 4.

Definition. Let $k\in\mathbf{N}$, $y\in\Lambda^{(k)}$, $t>0$. A (k, y, t)-admissible charge density ρ consists of:

(i) A charge density ρ localized on $B_k(y)$, i.e., $\rho:\mathbf{Z}^2\to\mathbf{R}$ with $\operatorname{supp}\rho\subset\bar{B}_k(y)$ and total charge $Q(\rho)=\sum_u\rho(u)\in\mathbf{Z}$, with $|\rho|=\sum_{u\in\mathbf{Z}^2}|\rho(u)|\geqslant 1$ unless $\rho\equiv 0$.

(ii) A complex-valued activity functional $z(\rho,\phi)=e^{\gamma(\rho,\phi)}$, where $\gamma(\rho,\phi)$ is a complex-valued real-analytic function of the real variables $\{\phi(u),\ u\in\bar{B}_k(y)\}$ such that (a) $\omega(\rho,\phi)=\Re\gamma(\rho,\phi)$ is even in ϕ, i.e., $\omega(\rho,-\phi)=\omega(\rho,\phi)$, and $\vartheta(\rho,\phi)=\Im\gamma(\rho,\phi)$ is odd in ϕ, i.e., $\vartheta(\rho,-\phi)=-\vartheta(\rho,\phi)$; (b) the following holds:

$$|z(\rho,\phi)|=e^{\omega(\rho,\phi)}\leqslant L_k^{-t}e^{-|\rho|/\log L_k} \tag{2.4}$$

and (c) for $n=1,2,\ldots$, let $\delta_j=\varepsilon_{x_j}^{(l_j)}\in\mathscr{F}_k$, $j=1,2,\ldots,n$; then

$$\left|\prod_{j=1}^{n}\left(\delta_j\cdot\frac{\partial}{\partial\phi}\right)\gamma(\rho,\phi)\right|\leqslant n!\,|\rho|\prod_{j=1}^{n}d(\rho,\delta_j) \tag{2.5}$$

where

$$\delta\cdot\frac{\partial}{\partial\phi}=\sum_u\delta(u)\frac{\partial}{\partial\phi(u)}$$

and

$$d(\rho, \varepsilon_u^{(l)}) = \begin{cases} \dfrac{L_k}{|y-u|_2} & \text{if } \bar{B}_k(y) \cap \left\{x; \dfrac{15}{6} L_l \leqslant |x-u|_2 \leqslant \dfrac{1}{6} L_{l+1}\right\} \neq \varnothing \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

Remark. If $\delta \in \mathscr{F}_k$, and ρ is (k, y, t)-admissible, then for any value of ϕ we have $\gamma(\rho, \phi + \zeta\delta)$ defined and analytic in ζ for $|\zeta| < d(\rho, \delta)^{-1}$.

Definition. By an admissible charge density we will mean a (k, y, t)-admissible charge density for some k, y, t.

Definition. Let $p > 2$ be fixed, $k \in \mathbf{N}$, $y \in \Lambda^{(k)}$, $r > 0$. A collection $\mathscr{N}_{(k,y,r)}$ of neutral [i.e., $Q(\rho) = 0$] admissible charge densities will be called a (k, y, r)-sparse neutral ensemble if:

(i) For $k = 1$, $\mathscr{N}_{(1,y,r)} = \varnothing$.

(ii) For $k = 2, 3, \ldots$, we have

$$\mathscr{N}_{(k,y,r)} = \left(\bigcup_{y' \in B_k^{(k-1)}(y)} \mathscr{N}_{(k-1,y',r)} \right) \bigcup \{\rho\}$$

where $\mathscr{N}_{(k-1,y',r)}$ is a $(k-1, y', r)$-sparse neutral ensemble and ρ is a $(k-1, y'', r)$-admissible neutral charge density for some $y'' \in B_k^{(k-1)}(y)$ with (2.4) replaced by

$$|z(\rho, \phi)| \leqslant \frac{8}{(\log 2)^4} L_{k-1}^{-r} e^{-|\rho|/\log L_{k-1}} \tag{2.7}$$

and

$$1 \leqslant |\rho| \leqslant (\log L_{k-1})^p \tag{2.8}$$

We let

$$F(\mathscr{N}_{(k,y,r)}; \phi) = \prod_{\rho \in \mathscr{N}_{(k,y,r)}} \{1 + e^{\omega(\rho,\phi)} \cos[\phi(\rho) + \vartheta(\rho, \phi]\}$$

Definition. Given $k \in \mathbf{N}$ and $r > 0$, a (k, r)-regular charge assignment is a collection $\{\mathscr{N}_{(k,y,r)}, \rho_y\}_{y \in \Lambda^{(k)}}$, where $\mathscr{N}_{(k,y,r)}$ is a (k, y, r)-sparse neutral ensemble and ρ_y is a $[k, y, t + 2(\alpha - 1)]$-admissible charge density, the ρ_y's having disjoint supports.

Definition. A (k, r)-regular partition function is a partition function of the form

$$Z_{(k,r)} = \int \prod_{y \in \Lambda^{(k)}} K_{(k,y,r)}(\phi) \, d\mu_\beta(\phi) \tag{2.9}$$

where

$$K_{(k,y,r)}(\phi) = F(\mathcal{N}_{(k,y,r)}; \phi)\{1 + e^{\omega(\rho_y,\phi)} \cos[\phi(\rho_y) + \vartheta(\rho_y, \phi)]\}$$

with $\{\mathcal{N}_{(k,y,r)}, \rho_y\}_{y \in \Lambda^{(k)}}$ being a (k, r)-regular charge assignment.

We have the following result.

Theorem 2.1. Let $1 < \alpha < 2$ and suppose $r > 0$ is such that

$$2\alpha \frac{\alpha - 1}{2 - \alpha} < r < \frac{\beta}{4\pi} - 2\alpha \tag{2.10}$$

Then, if the initial scale L_1 is sufficiently large, there exists $0 < \bar{z} = \bar{z}(\alpha, r, \beta, p, L_1)$ such that if the activity z is such that $0 < z < \bar{z}$, the Coulomb gas partition function Z_Λ can always be written as a convex combination of (k, r)-regular partition functions for any $k = 1, 2,..., N$.

Remark. Notice that (2.10) can be satisfied if and only if $\beta > 8\pi\alpha/(2 - \alpha)$.

Proof. The proof is by induction. The initial step, $k = 1$, follows immediately from (2.1) and (2.2).

The proof of the inductive step, as the proof of Theorem 3.1 in ref. 4, uses two basic lemmas. The first is just Lemma 2.1 of ref. 4, which was already used in the derivation of (2.1) and (2.2). We will now rewrite it in the form in which we will use it, for the reader's convenience (we just substitute e^{ω_i} for z_i and $\phi_i + \vartheta_i$ for ϕ_i).

Lemma 2.2. Let I be an index set with N elements, and let ω_i, ϑ_i, $\phi_i \in \mathbf{R}$ be given for each $i \in I$. Then

$$\prod_{i \in I} [1 + e^{\omega_i} \cos(\phi_i + \vartheta_i)] = \sum_{\sigma \in \mathscr{G}(I)} c_\sigma [1 + e^{\omega_\sigma} \cos(\phi_\sigma + \vartheta_\sigma)]$$

where $\mathscr{G}(I) = \{\sigma: I \to \{0, 1, -1\}; \sigma \text{ not identically zero}\}$,

$$\phi_\sigma = \sum_{i \in I} \sigma_i \phi_i$$

$$\vartheta_\sigma = \sum_{i \in I} \sigma_i \vartheta_i$$

$$\omega_\sigma = \sum_{i \in I} |\sigma_i| (\omega_i + \log b_N)$$

where b_N is a constant depending only on N given by

$$b_N = (2^{1/N} - 1)^{-1} \leqslant \frac{N}{\log 2}$$

and $0 < c_\sigma$, $\sum_{\sigma \in \mathscr{G}(I)} c_\sigma = 1$.

Now let $k \in \{1, 2, \ldots, N-1\}$ and let $\{\mathcal{N}_{(k,y,r)}, \rho_y\}_{y \in \Lambda^{(k)}}$ be a (k, r)-regular charge assignment. Let $Z_{(k,r)}$ be given by (2.9). As in ref. 4, pp. 147–148, we can use Lemma 2.2 to rewrite $Z_{(k,r)}$ as a convex combination of partition functions of the form

$$\int \prod_{u \in \Lambda^{(k+1)}} K^*_{(k+1,u,r)}(\phi)\, d\mu_\beta(\phi) \tag{2.11}$$

where for each $u \in \Lambda^{(k+1)}$

$$K^*_{(k+1,u,r)} = F(\tilde{\mathcal{N}}_{(k+1,u,r)}; \phi)\{1 + e^{\omega(\rho^*_u, \phi)} \cos[\phi(\rho^*_u) + \vartheta(\rho^*_u, \phi)]\}$$

where

$$\tilde{\mathcal{N}}_{(k+1,u,r)} = \bigcup_{y' \in B^{(k)}_{k+1}(u)} \mathcal{N}_{(k,y',r)} \tag{2.12}$$

is a $(k+1, u, r)$-sparse neutral ensemble and, for each $u \in \Lambda^{(k+1)}$, ρ^*_u is of the form

$$\rho^*_u = \sum_{y \in \bar{B}^{(k)}_{k+1}(u)} \sigma_y \rho_y \tag{2.13}$$

for some $\sigma \in \mathcal{G}(\bar{B}^{(k)}_{k+1}(u))$, and

$$\omega(\rho^*_u, \phi) = \sum_{y \in \bar{B}^{(k)}_{k+1}(u)} |\sigma_y|\, \omega(\rho_y, \phi) + c_u \tag{2.14}$$

$$\vartheta(\rho^*_u, \phi) = \sum_{y \in \bar{B}^{(k)}_{k+1}(u)} \sigma_y \vartheta(\rho_y, \phi) \tag{2.15}$$

where c_u is constant in ϕ,

$$\gamma(\rho^*_u, \phi) = \omega(\rho^*_u, \phi) + i\vartheta(\rho^*_u, \phi) \tag{2.16}$$

$$e^{\omega(\rho^*_u, \phi)} \leqslant \prod_{\substack{y \in B^{(k)}_{k+1}(u) \\ \sigma_y \neq 0}} \left[\frac{8}{(\log 2)^4} L_k^{-r} e^{-|\rho_y|/\log L_k}\right] \tag{2.17}$$

all the ρ^*_u, $u \in \Lambda^{(k+1)}$, have disjoint support, and we arranged this so that if for some $u_0 \in \Lambda^{(k+1)}$ we have $\rho^*_{u_0} = \rho_y$ for some $y \in \bar{B}^{(k)}_{k+1}(u_0)$, then $y \in B^{(k)}_{k+1}(u_0)$, and moreover,

$$B(y, \tfrac{1}{3}L_{k+1}) \cap \operatorname{supp} \rho^*_u = \varnothing \tag{2.18}$$

for $u \in \Lambda^{(k+1)}$, $u \neq u_0$.

To propagate our bounds on the activities to the next scale, we will need, in some cases, to extract a self-energy term as in refs. 3 and 4. This will be done by the following lemma, which improves on the energy estimates of previous approaches (compare with Lemma 3.3 in ref. 4; the crucial difference is that now we only require $\alpha > 1$ instead of $\alpha > 3/2$).

Lemma 2.3. Let $1 < \alpha < 2$, r satisfying (2.10), and let $\{\tilde{\mathcal{N}}_{(k+1,u,r)}, \rho_u^*\}_{u \in \Lambda^{(k+1)}}$ be given by (2.12)–(2.17). Suppose that for some $u_0 \in \Lambda^{(k+1)}$ we have $\rho_{u_0}^* = \rho_y$ for some $y \in B_{k+1}^{(k)}(u_0)$ and (2.18) holds. Then, we have, for any $\kappa > 0$,

$$\begin{aligned}&\int \{\exp[\gamma(\rho_{u_0}^*, \phi) + i\phi(\rho_{u_0}^*)]\}\, F(\tilde{\mathcal{N}}_{(k+1,u_0,r)}; \phi)\\ &\qquad \times \prod_{u \neq u_0} K_{(k+1,u,r)}^*(\phi)\, d\mu_\beta(\phi)\\ &\quad = Y^{(k)} \int \{\exp[\gamma(\bar{\rho}_{u_0}, \phi) + i\phi(\bar{\rho}_{u_0})]\}\, F(\tilde{\mathcal{N}}_{(k+1),u_0,r)}; \phi)\\ &\qquad \times \prod_{u \neq u_0} K_{(k+1,u_0,r)}^*(\phi)\, d\mu_\beta(\phi)\end{aligned} \tag{2.19a}$$

where $Y^{(k)}$ is a constant satisfying

$$\left(\frac{L_{k+1}}{15L_k}\right)^{-\kappa|q|} \leqslant Y^{(k)} \leqslant \left(\frac{L_{k+1}}{15L_k}\right)^{-\kappa(|q| - \kappa(\pi/\beta)(1+a))} \tag{2.19b}$$

where $q = Q(\rho_{u_0}^*)$, $\bar{\rho}_{u_0}$ is a charge density with support on $B(y, \frac{1}{3}L_{k+1})$ such that $Q(\bar{\rho}_{u_0}) = q$,

$$c'' \frac{\kappa}{\beta} \log L_k \leqslant |\bar{\rho}_{u_0}| - |\rho_{u_0}^*| \leqslant c' \frac{\kappa}{\beta} \log L_k$$

with $0 < c'' \leqslant c' < \infty$, $a = a(\alpha, L_1) > 0$ with $\lim_{L_1 \to \infty} a = 0$, and $\gamma(\bar{\rho}_{u_0}, \phi)$ is a complex-valued real-analytic function of the real variables $\{\phi(x); x \in \bar{B}_{k+1}(u_0)\}$, with $\Re\gamma(\bar{\rho}_{u_0}, \phi)$ even in ϕ, $\Im\gamma(\bar{\rho}_{u_0}, \phi)$ odd in ϕ, and

$$|\Re(\gamma(\bar{\rho}_{u_0}, \phi) - \gamma(\rho_{u_0}^*, \phi))| \leqslant \kappa^2 b\pi \log \frac{L_{k+1}}{15L_k} \tag{2.20}$$

for some $b = b(\alpha, r, L_1)$ with $\lim_{L_1 \to \infty} b = 0$. Moreover, for all $n = 1, 2, \ldots$, $\delta_j \in \mathscr{F}_{k+1}$, $j = 1, \ldots, n$,

$$\left|\prod_{j=1}^{n} \left(\delta_j \cdot \frac{\partial}{\partial\phi}\right) \gamma(\bar{\rho}_{u_0}, \phi)\right| \leqslant n! \left(\left[\frac{L_k}{L_{k+1}}\right]^n |\rho_{u_0}^*| + \kappa\tilde{C}\right) \prod_{j=1}^{n} d(\bar{\rho}_{u_0}, \delta_j) \tag{2.21}$$

where $\tilde{C} = \tilde{C}(L_1, \alpha, r, p)$ with $\lim_{L_1 \to \infty} \tilde{C} = 0$.

Lemma 2.3 will be proven in Section 4.

Remark. Lemma 2.3 is true also if we take the complex conjugation of both sides in (2.19a). We could thus replace $e^{\gamma+i\phi}+e^{\bar{\gamma}-i\phi}$ by $2e^{\Re\gamma}\cos(\phi+\Im\gamma)$. We have stated Lemma 2.3 in this form for convenience in further applications.

We can now finish the proof of Theorem 2.1. Let us fix $u\in\Lambda^{(k+1)}$, and let $\sigma_y\in\mathscr{G}(\bar{B}^{(k)}_{k+1}(u))$ be he one in (2.13). We consider several cases:

(i) $\sum_y |\sigma_y|\geqslant 2$.

In this case we define $\mathscr{N}_{(k+1,u,r)}=\tilde{\mathscr{N}}_{(k+1,u,r)}$ and $\rho_u=\rho_u^*$. We must prove ρ_u is $[k+1, u, r+2(\alpha-1)]$-admissible.

From (2.17) we have

$$|z(\rho_u,\phi)|\leqslant\frac{1}{L_{k+1}^{r+2(\alpha-1)}}e^{-|\rho_u|/\log L_{k+1}}$$

if $r>2\alpha(\alpha-1)/(2-\alpha)$ and L_1 is sufficiently large, since the $\{\rho_y\}_{y\in\Lambda^{(k)}}$ have disjoint supports so

$$|\rho_u|=\sum_{y:\sigma_y\neq 0}|\rho_y| \tag{2.22}$$

It remains to prove (2.5). Since each $\omega(p,\phi)$ is even in ϕ and each $\vartheta(\rho,\phi)$ is odd, we have from (2.14) and (2.15) that

$$\gamma(\rho_u,\phi)=\sum_{y\in\bar{B}^{(k)}_{k+1}(u)}|\sigma_y|\,\gamma\left(\rho_y,\frac{\sigma_y}{|\sigma_y|}\phi\right)+c_u \tag{2.23}$$

Now let $\delta_1,\ldots,\delta_n\in\mathscr{F}_{k+1}$. It follows from (2.23), (2.5), and (2.22) that

$$\left|\prod_{j=1}^{n}\left(\delta_j\cdot\frac{\partial}{\partial\phi}\right)\gamma(\rho_u,\phi)\right|\leqslant n!\sum_{y:\sigma_y\neq 0}|\rho_y|\prod_{j=1}^{n}d(\rho_y,\delta_j)\leqslant n!\,|\rho_u|\prod_{j=1}^{n}d(\rho_u,\delta_j)$$

(ii) $\sum_y|\sigma_y|=1$.

Here we must consider three subcases:

(iia) $|\rho_u^*|\geqslant(\log L_k)^p$

We let $\mathscr{N}_{(k+1,u,r)}=\tilde{\mathscr{N}}_{(k+1,u,r)}$ and $\rho_u=\rho_u^*$.

Then (2.4) follows for ρ_u [in the $(k+1)$th scale] from (2.17), since if L_1 is sufficiently large, we have

$$(\log L_1)^{p-2}>\alpha(r+2\alpha)$$

Thus, ρ_u is $[k+1, u, r+2(\alpha-1)]$-admissible.

(iib) $|\rho_u^*|<(\log L_k)^p$ and $Q(\rho_u^*)=0$.

Then $\mathcal{N}_{(k+1,u,r)} = \tilde{\mathcal{N}}_{(k+1,u,r)} \cup \{\rho_u^*\}$ is a $(k+1, u, r)$-sparse neutral ensemble. We take $\rho_u \equiv 0$.

(iic) $|\rho_u^*| < (\log L_k)^p$ and $Q(\rho_u^*) \neq 0$.

(2.11) can be rewritten as

$$\int (1 + \tfrac{1}{2}\{\exp[\gamma(\rho_u^*, \phi) + i\phi(\rho_u^*)] + \exp[\bar{\gamma}(\rho_u^*, \phi) - i\phi(\rho_u^*)]\})$$

$$\times F(\tilde{\mathcal{N}}_{(k+1,u,r)}; \phi) \prod_{\substack{u' \in \Lambda^{(k+1)} \\ u' \neq u}} K^*_{(k+1,u',r)}(\phi)\, d\mu_\beta(\phi) \tag{2.24}$$

We apply Lemma 2.3 to (2.24), replacing ρ_u^* by $\bar{\rho}_u$. We take $\rho_u = \bar{\rho}_u$, $\gamma(\rho_u, \phi) = \gamma(\bar{\rho}_u, \phi) + \log Y^{(k)}$. Using (2.17), (2.19b), and (2.20), we get

$$|z(\rho_u, \phi)| \leqslant \frac{8}{(\log 2)^4} L_k^{-r} \left(\exp - \frac{|\rho_u^*|}{\log L_k}\right) \left(\frac{L_{k+1}}{15 L_k}\right)^{-\kappa(1 - \kappa(\pi/\beta)(1 + a + b\beta))}$$

We choose

$$\kappa = \left[\frac{2\pi}{\beta}(1 + a + b\beta)\right]^{-1}$$

and recall $0 \leqslant |\bar{\rho}_u| - |\rho_u^*| \leqslant c'(\kappa/\beta) \log L_k$.

Thus, if

$$r < \frac{\beta}{4\pi(1 + a + b\beta)} - 2\alpha$$

we have

$$|z(\rho_u, \phi)| \leqslant L_{k+1}^{-(r + 2(\alpha - 1))} e^{-|\rho_u|/\log L_{k+1}}$$

if L_1 is sufficiently large.

In view of (2.21), we can conclude that ρ_u is $[k+1, u, r + 2(\alpha - 1)]$-admissible if L_1 is sufficiently large. We take $\mathcal{N}_{(k+1,u,r)} = \tilde{\mathcal{N}}_{(k+1,u,r)}$.

This concludes the proof of Theorem 2.1.

3. POWER-LAW FALLOFF

We start by studying the external charges partition function $Z_{\xi,\Lambda}(x)$ given by (1.2). We want to prove the analogue of Theorem 2.1 for it. The extension of the treatment given to the partition function Z_Λ in Section 2 to $Z_{\xi,\Lambda}(x)$ is similar to the extension in ref. 4. We shall skip details when they are essentially done in ref. 4.

Given $\xi \in \mathbf{R}$, we let $\eta = \text{dist}(\xi, \mathbf{Z})$ and $\bar{\eta} = \text{dist}(\xi, \mathbf{Z}\backslash\{0\})$. We fix $x \in \mathbf{Z}^2$, and choose N_0 such that $\frac{1}{2}L_{N_0} < |x|_\infty < \frac{1}{2}L_{N_0+1}$. Without loss of generality we take $N_0 > 1$.

As in (2.1), the external charges partition function can be written as a convex combination of expansions of the form

$$\int e^{i\xi(\phi(0)-\phi(x))} \prod_{y \in \Lambda^{(1)}} [1 + z_y \cos \phi(\rho_y)]\, d\mu_\beta(\phi) \tag{3.1}$$

where $\{(\rho_y, z_y);\ y \in \Lambda^{(1)}\}$ are the same as in (2.1) and (2.2). This is the initial step in the inductive procedure we will now describe.

We have two distinguished sites, 0 and x. To keep track of these sites, we introduce the following notation: given $y \in \Lambda$, we denote by y_k the unique point in $\Lambda^{(k)}$ such that $y \in B_k(y_k)$. Notice that $0_k = 0$ for all k. At scale $k = 1, 2, \dots, N_0$ we will have two distinguished sites in $\Lambda^{(k)}$: 0 and x_k. For $k \geq N_0 + 1$ we have $x_k = 0$.

As in ref. 4, at the squares $B_k(0)$ and $B_k(x_k)$ we will replace the previously unique charge density ρ by two charges densities, ρ^+, ρ^-, with $Q(\rho^+) = Q(\rho^-)$; terms of the form $\{1 + e^{\omega(\rho,\phi)} \cos[\phi(\rho) + \vartheta(\rho, \phi)]\}$ are replaced by terms of the form

$$1 + \tfrac{1}{2}[e^{\gamma(\rho^+,\phi) + i\phi(\rho^+)} + e^{\bar{\gamma}(\rho^-,\phi) - i\phi(\rho^-)}]$$

where, as before, $\gamma(\rho^\pm, \phi) = \omega(\rho^\pm, \phi) + i\vartheta(\rho^\pm, \phi)$. Here $\bar{\gamma}$ is the complex conjugate of γ.

Definition. Let $k \in \mathbf{N}$, $y \in \Lambda^{(k)}$, $r > 0$. A pair (ρ^+, ρ^-) is a (k, y, s)-admissible pair of charge densities if:

(i) $\rho^\pm = \rho \pm \sigma$, where ρ and σ are charge densities with support in $\hat{B}_k(y) \equiv B(y, (10/3)L_k)$, $Q(\rho^+) = Q(\rho^-) \in \mathbf{Z}$ [so $Q(\sigma) = 0$], $|\rho| \geq 1$ unless $\rho \equiv 0$, and

$$|\sigma| \leq \frac{c'}{18\pi} \frac{\alpha^k - 1}{\alpha - 1} \log L_k$$

with $0 < c' < \infty$ given by (4.23).

(ii) $\rho^\pm$ have activity functionals $z(\rho^\pm, \phi) = e^{\gamma(\rho^\pm,\phi)}$ with $\gamma(\rho^\pm, \phi)$ being a complex-valued real-analytic function of $\{\phi(u);\ u \in \hat{B}_k(y)\}$ such that (a) $\omega(\rho^\pm, \phi) = \Re\gamma(\rho^\pm, \phi)$ is even in ϕ and $\vartheta(\rho^\pm, \phi) = \Im\gamma(\rho^\pm, \phi)$ is odd in ϕ; (b)

$$|z(\rho^\pm, \phi)| \leq L_k^{-s} e^{\langle \rho^\pm \rangle / \log L_k} \tag{3.2}$$

where $\langle \rho^{\pm} \rangle = |\rho| + |\sigma|$ (notice we allow $\operatorname{supp} \rho \cap \operatorname{supp} \sigma \neq \varnothing$); and (c)

$$\left| \prod_{j=1}^{n} \left(\delta_j \cdot \frac{\partial}{\partial \phi} \right) \gamma(\rho^{\pm}, \phi) \right| \leqslant n! \, \langle \rho^{\pm} \rangle \prod_{j=1}^{n} d(\rho^{\pm}, \delta_j)$$

with $\delta_1, ..., \delta_n \in \mathscr{F}_k$.

Definition. Let $k = 1, 2, ..., N$, $y_k = 0$ or x_k, $r > s > 0$. A collection $\mathscr{E}_{(k, y_k, r, s)}$ of neutral weighed charge densities will be called a (k, y_k, r, s)-sparse modified neutral ensemble if:

(i) For $k = 1$, $\mathscr{E}_{(1, y_1, r, s)} = \varnothing$.

(ii) For $k = 2, ..., N_0$,

$$\mathscr{E}_{(k, y_k, r, s)} = \bigcup_{\substack{u \in B_k^{(k-1)}(y_k) \\ u \neq y_{k-1}}} \mathscr{N}_{(k-1, u, r)} \bigcup \mathscr{E}_{(k-1, y_{k-1}, r, s)} \bigcup \{(\rho^+, \rho^-)\}$$

(iii) For $k = N_0 + 1$

$$\mathscr{E}_{(N_0+1, 0, r, s)} = \bigcup_{\substack{u \in B_{N_0+1}^{(N_0)}(0) \\ u \neq 0, x_{N_0}}} \mathscr{N}_{(N_0, u, r)} \bigcup_{y' = 0, x_{N_0}} \mathscr{E}_{(N_0, y', r, s)} \bigcup \{(\rho^+, \rho^-)\}$$

(iv) For $k > N_0 + 1$, we have as in (ii), where $\mathscr{N}_{(k-1, u, r)}$ is a $(k-1, u, r)$-sparse neutral ensemble, $\mathscr{E}_{(k-1, y, r, s)}$ is a $(k-1, y, r, s)$-sparse modified neutral ensemble, and (ρ^+, ρ^-) is a $(k-1, y', s)$-admissible pair of neutral charge densities for some $y' \in B_k^{(k-1)}(y_k)$ with (3.2) replaced by

$$|e^{\gamma(\rho^{\pm}, \phi)}| \leqslant \frac{2^3 \cdot 3}{(\log 2)^4} L_{k-1}^{-s}$$

and $\rho^{\pm} = \rho \pm \sigma$ with ρ satisfying (2.8).

Notice that $\mathscr{E}_{(k, y, r, s)}$ consists of charge densities ρ and of pairs of charge densities (ρ^+, ρ^-).

We let

$$G(\mathscr{E}_{(k, y, r, s)}; \phi) = \prod_{\rho \in \mathscr{E}_{(k, y, r, s)}} \{1 + e^{\omega(\rho, \phi)} \cos[\phi(\rho) + \vartheta(\rho, \phi)]\} \times \prod_{(\rho^+, \rho^-) \in \mathscr{E}_{(k, y, r, s)}} H((\rho^+, \rho^-), \phi)$$

where

$$H((\rho^+, \rho^-), \phi) = 1 + \tfrac{1}{2}(e^{\gamma(\rho^+, \phi) + i\phi(\rho^+)} + e^{\bar{\gamma}(\rho^-, \phi) - i\phi(\rho^-)})$$

Definition. Let $k=1,\ldots,N$, $r>s>0$. A (k,r,s)-modified charge assignment is a collection

$$\{\mathcal{N}_{(k,y,r)}, \rho_y\}_{y\in\Lambda^{(k)}\setminus\{0,x_k\}} \cup \{\mathcal{E}_{(k,y,r,s)}, (\rho_y^+, \rho_y^-)\}_{y=0,x_k}$$

where, for each $y\in\Lambda^{(k)}\setminus\{0,x_k\}$, $\mathcal{N}_{(k,y,r)}$ is a (k,y,r)-sparse neutral ensemble and ρ_y is a $[k,y,r+2(\alpha-1)]$-admissible charge density, and for $y=0, x_k$, $\mathcal{E}_{(k,y,r,s)}$ is a (k,y,r,s)-sparse modified neutral ensemble and (ρ^+,ρ^-) is a (k,y,s)-admissible pair of charge densities. In addition, the ρ_y's and (ρ_y^+,ρ_y^-) have disjoint support, where the support of (ρ^+,ρ^-) is the union of the supports of ρ^+ and ρ^-.

Definition. A (k,r,s)-regular external charges partition function is an expression of the form

$$Z_{\xi,\Lambda}^{(k,r,s)}(x)=\int e^{i\xi\phi(w_k)} \prod_{y\in\Lambda^{(k)}\setminus\{0,x_k\}} K_{(k,y,r)}(\phi) \times \prod_{y=0,x_k} W_y^{(k)}(\phi)\, R_{(k,y,r,s)}(\phi)\, d\mu_\beta(\phi) \tag{3.3a}$$

where $K_{(k,y,r)}(\phi)$ is as in (2.9) and

$$R_{(k,y,r,s)}(\phi)=G(\mathcal{E}_{(k,y,r,s)};\phi)\, H((\rho_y^+,\rho_y^-),\phi)$$

with

$$\{\mathcal{N}_{(k,y,r)}, \rho_y\}_{y\in\Lambda^{(k)}\setminus\{0,x_k\}} \cup \{\mathcal{E}_{(k,y,r,s)}, (\rho_y^+, \rho_y^-)\}_{y=0,x_k}$$

being a (k,r,s) modified charge assignment; w_k is the modified external charge density; for $k=1,2,\ldots,N_0$, $w_k=w_k^{(0)}+w_k^{(x)}$, where $w_k^{(0)}$, $w_k^{(x)}$ are external charge densities with disjoint supports contained in $B_k(0)$, $B_k(x_k)$, respectively, and $Q(w_k^{(0)})=-Q(w_k^{(x)})=1$; for $k\geqslant N_0+1$, w_k is a neutral charge density localized in $B_k(0)$. In addition, $W_y^{(k)}(\phi)=\exp[\gamma(\xi w_k^{(y)},\phi)]$, where $\gamma(\xi w_k^{(y)};\phi)$ is a complex-valued real analytic function of $\{\phi(u); u\in\bar{B}_k(y)\}$ with even real part and odd imaginary part, satisfying (2.5), and such that:

(i) $W_y^{(1)}(\phi)\equiv 1$.

(ii) We have

$$\left(\frac{L_k}{15L_{k-1}}\right)^{-\beta\delta\eta^2/8\pi} |W_y^{(k-1)}(\phi)| \leqslant |W_y^{(k)}(\phi)| \leqslant \left(\frac{L_k}{15L_{k-1}}\right)^{-\beta\delta\eta^2/10\pi} |W_y^{(k-1)}(\phi)| \tag{3.3b}$$

for $k=1,\ldots,N_0$, where $\delta=1/(1+a+\beta b)$ with a, b given in Lemma 2.3.

(iii) $W_0^{(k)}(\phi)=W_0^{(N_0)}(\phi)\,W_x^{(N_0)}(\phi)$ for $k\geqslant N_0+1$.

The extension of Theorem 2.1 is as follows.

Theorem 3.1. Let $1<\alpha<2$ and r satisfying (2.10). There exists $\delta=\delta(\alpha, r, \beta, L_1)$ with $\lim_{L_1\to\infty}\delta=1$, such that, if

$$0<s<\min\left\{\frac{1}{\alpha}\left(r-\frac{\beta(\alpha-1)\,\delta\eta^2}{8\pi}\right), \frac{\beta\delta(\bar{\eta}^2-\eta^2/2)}{4\pi}\right\} \tag{3.4}$$

and the initial scale L_1 is sufficiently large, there exists $0<z(\alpha, r, \beta, p, L_1)$ such that if the activity z is such that $0<z<z(\alpha, r, \beta, p, L_1)$, the Coulomb-gas external-charges partition function $Z_{\xi,\Lambda}(x)$ can always be written as a convex combination of (k, r, s)-regular partition functions for any $k=1, 2,..., N$.

Proof. Theorem 3.1 is proven by induction. The initial step, $k=1$, follows from (3.1). The inductive step proceeds as in the proof of Theorem 4.3 in ref. 4, using Lemma 2.3 instead of Lemma 3.3 in ref. 4. We present here the induction for $k=1, 2,..., N_0-1$, the modifications for $k\geqslant N_0$ being just as in ref. 4.

So, let $k\in\{1,..., N_0-1\}$, and let $Z_{\xi,\Lambda}^{(k,r,s)}(x)$ be given by (3.3a) with $r>s>0$ satisfying (2.10) and (3.4). Proceeding as in the proof of Theorem 4.3 in ref. 4, one can write this as a convex combination of expressions of the form

$$\int e^{i\xi\phi(w_k)} \prod_{u\in\Lambda^{(k+1)}\setminus\{0, x_{k+1}\}} K_{(k+1,u,r)}(\phi) \prod_{y=0,x_{k+1}} W_y^{(k)}(\phi)\,R^*_{(k+1,y,r,s)}(\phi)\,d\mu_\beta(\phi) \tag{3.5}$$

where $K_{(k+1,u,r)}(\phi)$, $u\in\Lambda^{(k+1)}\setminus\{0, x_{k+1}\}$ is as in (2.9) (at scale $k+1$; here we used the proof of Theorem 2.1),

$$R^*_{(k+1,y,r,s)}=G(\tilde{\mathscr{E}}_{(k+1,y,r,s)};\phi)\,H((\rho_y^{*+}, \rho_y^{*-}), \phi)$$

with

$$\tilde{\mathscr{E}}_{(k+1,y,r,s)}=\bigcup_{\substack{u\in B_{k+1}^{(k)}(y)\\ u\neq y_k}}\mathscr{N}_{(k,u,r)}\cup\mathscr{E}_{(k,y_k,r,s)}$$

being a $(k+1, y, r, s)$-sparse modified neutral ensemble, and $(\rho_y^{*+}, \rho_y^{*-})$ is a pair of charges with support in $\hat{B}_{k+1}(y_{k+1})$ of the form $\rho_y^{*\pm}=\rho_y^*+\tau_2\rho_{y_k}^{\pm}$, with complex-valued activities functionals $z(\rho_y^{*\pm}, \phi)=\exp[\gamma(\rho_y^{*\pm}, \phi)]$, $\gamma(\rho_y^{*\pm}, \phi)$ being a complex-valued real-analytic function of the variables $\{\phi(u),\ u\in \hat{B}_{k+1}(y_{k+1})\}$ such that its real part $\omega(\rho_y^{*\pm}, \phi)$ is even in ϕ and its imaginary part $\vartheta(\rho_y^{*\pm}, \phi)$ is odd in ϕ,

$$|z(\rho_y^{*\pm}, \phi)| \leqslant \left[\frac{2^4}{(\log 2)^5}\frac{1}{L_k^r}\right]^{\tau_1}\left[\left(\frac{2}{\log 2}\right)^4 |z(\rho_{y_k}^{\pm}, \phi)|\right]^{\tau_2} e^{-\langle\rho_y^{*\pm}\rangle/\log L_k} \tag{3.6}$$

for some $\tau_1, \tau_2 = 0$ or 1 with $\tau_1+\tau_2\neq 0$ (notice that we actually have $\tau_2=0$ or 1 in Lemma 4.1 of ref. 4).

Remark. Lemma 2.1 was stated in the form in which it was used in Section 2. In this section we will use it in (3.5) with $\rho_{u_0}^*$ replaced by either $\xi w_k^{(y)}$ or $\xi w_k^{(y)} \pm \rho_{y_k}^{*\pm}$; the proof is still valid with the same conclusions. Notice that it follows from (2.3) that the imaginary shift in the proof of Lemma 2.3 is constant on the support of $\rho_{y_k}^{\pm}$, so their presence does not affect Lemma 2.3.

We consider several cases:

(i) $\tau_1\neq 0$. We define $\mathscr{E}_{(k+1,y,r,s)} = \tilde{\mathscr{E}}_{(k+1,y,r,s)}$, and look separately at each factor of

$$\{\exp[i\xi\phi(w_k^{(y)})]\}\ W_y^{(k)}(1+\tfrac{1}{2}\{\exp[\gamma(\rho_y^{*+}, \phi)+i\phi(\rho_y^{*+})] + \exp[\bar{\gamma}(\rho_y^{*-}, \phi)-i\phi(\rho_y^{*-})]\}) \tag{3.7}$$

We use Lemma 2.3 for the first factor to replace $w_k^{(y)}$ by

$$w_{k+1}^{(y)} = \overline{w_k^{(y)}} \equiv \frac{\overline{\xi w_k^{(y)}}}{\xi}$$

and $W_y^{(k)}$ by $W_y^{(k+1)}$; we choose $\kappa = \eta^2\beta\delta/9\pi\xi$. It follows from (2.19b) and (2.20) that (3.3b) is satisfied; (2.5) follows from (2.21). We then see that (3.7) can be replaced by

$$\{\exp[i\xi\phi(w_{k+1}^{(y)})]\}\ W_y^{(k+1)}(1+\tfrac{1}{2}\{\exp[\gamma(\tilde{\rho}_y^{+}, \phi)+i\phi(\tilde{\rho}_y^{+})] + \exp[\bar{\gamma}(\tilde{\rho}_y^{-}, \phi)-i\phi(\tilde{\rho}_y^{-})]\}) \tag{3.8}$$

where

$$\tilde{\rho}_y^{\pm} = \rho_y^{*\pm} \pm \xi Q(w_k^{(y)})(w_k^{(y)} - w_{k+1}^{(y)})$$

so

$$\begin{aligned}\langle\rho_y^{*\pm}\rangle &\geqslant \langle\tilde{\rho}_y^{\pm}\rangle - \xi\,|w_k^{(y)} - w_{k+1}^{(y)}| \\ &\geqslant \langle\tilde{\rho}_y^{\pm}\rangle - \frac{c'\delta\eta^2}{9\pi\xi}\log L_k \\ &\geqslant \langle\tilde{\rho}_y^{\pm}\rangle - \frac{c'\delta}{18\pi}\log L_k\end{aligned}$$

by Lemma 2.3, and

$$\gamma(\tilde{\rho}_y^{\pm}, \phi) = \gamma(\rho_y^{*\pm}, \phi) + \log \frac{W_y^{(k)}}{W_y^{(k+1)}}$$

It follows that

$$|e^{\gamma(\tilde{\rho}_y^{\pm}, \phi)}| \leqslant \left(\frac{L_{k+1}}{15L_k}\right)^{\beta\delta\eta^2/8\pi} \frac{2^4}{(\log 2)^5} \frac{1}{L_k^r} e^{c'\delta/18\pi} e^{-\langle\tilde{\rho}_y^{\mp}\rangle/\log L_k}$$
$$\leqslant \left(\frac{1}{L_{k+1}}\right)^s e^{-\langle\tilde{\rho}_y^{\pm}\rangle/\log L_{k+1}} \tag{3.9}$$

if $s < (1/\alpha)[r - \beta\delta(\alpha - 1)\eta^2/8\pi]$ and L_1 sufficiently large. So $(\tilde{\rho}_y^+, \tilde{\rho}_y^-)$ is a $(k+1, y_{k+1}, s)$-admissible pair of charge densities.

(ii) $\tau_1 = 0$, so $\rho_y^{*\pm} = \rho_{y_k}^{\pm} = \rho_{y_k} \pm \sigma_{y_k}$. We consider three subcases:

(iia) $|\rho_{y_k}| \geqslant (\log L_k)^p$. We set $\mathscr{E}_{(k+1, y, r, s)} = \tilde{\mathscr{E}}_{(k+1, y, r, s)}$, and as before apply Lemma 2.3 to the first factor in (3.7), obtaining (3.8). The only difference is that we do not have (3.9), but it follows from (3.6) and (3.3b) that

$$|e^{\gamma(\tilde{\rho}_y^{\pm}, \phi)}| \leqslant \left(\frac{L_{k+1}}{15L_k}\right)^{\beta\delta\eta^2/8\pi} \left(\frac{2}{\log 2}\right)^4$$
$$\times L_k^{-s} \left(\frac{L_{k+1}}{L_k}\right)^{-(1/\alpha)(\log L_k)^{p-2}} e^{c'\delta/18\pi} e^{-\langle\tilde{\rho}_y^{\pm}\rangle/\log L_k}$$
$$\leqslant L_{k+1}^{-s} e^{-\langle\tilde{\rho}_y^{\pm}\rangle/\log L_{k+1}}$$

if L_1 is sufficiently large, so

$$(\log L_1)^{p-2} > \alpha\left(s + \frac{\beta\delta}{32\pi}\right) \geqslant \alpha\left(s + \frac{\beta\delta}{8\pi}\eta^2\right)$$

for all η since $\eta^2 \leqslant 1/4$.

(iib) $|\rho_{y_k}| < (\log L_k)^p$ and $Q(\rho_{y_k}) = 0$. We apply Lemma 2.3 to all factors in (3.7), choosing $\kappa = \eta^2\beta\delta/9\pi\xi$, and we notice that (3.7) can be written as [see (4.22) for a similar argument]

$$\{\exp[i\xi\phi(w_{k+1}^{(y)})]\}\, W_y^{(k+1)}(1 + \tfrac{1}{2}\{\exp[\gamma(\rho_y^{*+}, \phi) + i\phi(\rho_y^{*+})]$$
$$+ \exp[\bar{\gamma}(\bar{\rho}_y^{*-}, \phi) - i\phi(\rho_y^{*-})]\})$$

It follows from (3.6) that

$$\mathscr{E}_{(k+1, y, r, s)} = \tilde{\mathscr{E}}_{(k+1, y, r, s)} \bigcup \{(\rho_y^{*+}, \rho_y^{*-})\}$$

is a $(k+1, y, r, s)$-sparse modified neutral ensemble.

(iic) $|\rho_{y_k}| < (\log L_k)^p$ and $Q(\rho_{y_k}) \neq 0$. We take $\mathscr{E}_{(k+1, y, r, s)} = \tilde{\mathscr{E}}_{(k+1, y, r, s)}$, and apply Lemma 2.3 to each term in (3.7), replacing it by (3.8), where $w_{k+1}^{(y)}$, $W_y^{(k+1)}$ are as above and

$$\tilde{\rho}_y^{\pm} = \overline{\rho_y^{*\pm} \pm \xi w_k^{(y)}} \mp \xi w_{k+1}^{(y)}$$

$$\gamma(\tilde{\rho}_y^{\pm}, \phi) = \gamma(\overline{\rho_y^{*\pm}}, \phi) + \log \frac{W_y^{(k)}(\phi)}{W_y^{(k+1)}(\phi)} + \log Y^{(k)}$$

so

$$|e^{\gamma(\tilde{\rho}_y^{\pm}\, \phi)}| \leqslant \left(\frac{L_{k+1}}{15 L_k}\right)^{\beta\delta\eta^2/8\pi} \left(\frac{2}{\log 2}\right)^4 L_k^{-s} \left(\frac{L_{k+1}}{15 L_k}\right)^{-\beta\delta\eta^2/4\pi} e^{-\langle \rho_y^{*\pm} \rangle/\log L_k}$$

[Notice that we chose κ in Lemma 2.3, for the second and third terms in (3.7), to be $\kappa = \beta\delta\bar{\eta}/2\pi$.]

Now, by Lemma 2.3,

$$\begin{aligned} \langle \tilde{\rho}_y^{\pm} \rangle &\leqslant \langle \rho_y^{*\pm} \pm \xi(w_k^{(y)} - w_{k+1}^{(y)}) \rangle + \frac{c'\delta\bar{\eta}}{2\pi} \log L_k \\ &\leqslant \langle \rho_y^{*\pm} \rangle + \frac{c'\delta\eta^2}{9\pi\xi} \log L_k + \frac{c'\delta\bar{\eta}}{2\pi} \log L_k \\ &\leqslant \langle \rho_y^{*\pm} \rangle + \frac{5c'\delta}{9\pi} \log L_k \end{aligned}$$

Thus

$$|e^{\gamma(\tilde{\rho}_y^{\pm}\, \phi)}| \leqslant L_{k+1}^{-s} e^{-\langle \tilde{\rho}_y^{\pm} \rangle/\log L_{k+1}}$$

if $s < (\beta\delta/4\pi)(\bar{\eta}^2 - \eta^2/2)$ and L_1 sufficiently large.

This completes the proof of Theorem 3.1 for $k \leqslant N_0 - 1$.

We can now prove Theorem 1.1. We follow the proof of Theorems 1.1 and 1.2 in ref. 4. Let $x \in \mathbf{Z}^2$ with $L_{N_0} < 2|x|_\infty < L_{N_0+1}$, $N_0 \geqslant 1$, and let $\Lambda = B(0, R)$ with $L_{N-1} < R \leqslant L_N$, $N > N_0$. For any $\beta > 8\pi$, we pick $1 < \alpha < 2$, L_1 sufficiently large, and $r > 0$ such that

$$\frac{2\alpha(\alpha - 1)}{2 - \alpha} < r < \frac{\beta\delta}{4\pi} - 2\alpha \tag{3.10}$$

Notice that (3.10) requires

$$\beta > \frac{8\pi}{\delta} \frac{\alpha}{2 - \alpha}$$

which can always be satisfied for $\beta > 8\pi$ by picking α sufficiently close to one and L_1 sufficiently large, since $\lim_{L_1 \to \infty} \delta = 1$. We take L_1 large enough so that Theorem 3.1 holds. Then we have, for any $0 < z < z(L_1, \alpha)$, that the external charges correlation function $G_{\xi, \Lambda}(x)$ can be written as

$$G_{\xi, \Lambda}(x) = \frac{\sum_{\gamma \in \mathscr{F}} c_\gamma Z_{\xi, \Lambda, \gamma}^{(N,r,s)}(x)}{\sum_{\gamma \in \mathscr{F}} c_\gamma Z_{0, \Lambda, \gamma}^{(N,r,s)}} = \sum_{\gamma \in \mathscr{F}} d_\gamma \frac{Z_{\xi, \Lambda, \gamma}^{(N,r,s)}(x)}{Z_{0, \Lambda, \gamma}^{(N,r,s)}} \tag{3.11}$$

where c_γ, $d_\gamma > 0$, $\sum_{\gamma \in \mathscr{F}} c_\gamma = \sum_{\gamma \in \mathscr{F}} d_\gamma = 1$, where for each $\gamma \in \mathscr{F}$, $Z_{\xi, \Lambda, \gamma}^{(N,r,s)}(x)$ is an (N, r, s)-regular external charges partition function and $Z_{0, \Lambda, \gamma}^{(N,r,s)}$ is the same expression with $\xi = 0$. Here r, s are chosen satisfying (2.10) and (3.4), which can be done because of (3.10).

We must first show that (3.11) is well defined. Notice that $\xi \to 0$ implies $\eta \to 0$ and $\bar{\eta} \to 1$. Thus, if s satisfies (3.4) for any ξ, it also satisfies (3.4) for $\xi = 0$, and Theorem 3.1 also applies to $\xi = 0$. Moreover, for any $\gamma \in \mathscr{F}$ the partition function $Z_{0, \Lambda, \gamma}^{(N,r,s)}$ has the following properties:

(i) $W_0^{(N)}(\phi) \equiv 1$.

(ii) For all $k = 1, \ldots, N$ and $y = 0$, x, $(\rho_{y_k}^+, \rho_{y_k}^-) \in \mathscr{E}_{(k, y_k, r, s)}^{\gamma}$ satisfy $\rho_{y_k}^+ = \rho_{y_k}^- = \rho_{y_k}$ and $\gamma(\rho_{y_k}^+, \phi) = \gamma(\rho_{y_k}^-, \phi)$, so

$$\begin{aligned} 1 + \tfrac{1}{2}\{\exp[\gamma(\rho_y^+, \phi) + i\phi(\rho_{y_k}^+)] + \exp[\bar{\gamma}(\rho_{y_k}^-, \phi) - i\phi(\rho_{y_k}^-)]\} \\ = 1 + e^{\omega(\rho_{y_k}, \phi)} \cos[\phi(\rho_{y_k}) + \vartheta(\rho_{y_k}, \phi)] \end{aligned} \tag{3.12}$$

where ρ_{y_k} is a (k, y_k, s)-admissible charge density.

Thus we have

$$1 + e^{\omega(\rho_{y_k}, \phi)} \cos[\phi(\rho_{y_k}) + \vartheta(\rho_{y_k}, \phi)] \geqslant 1 - \frac{c}{L_k^s} \tag{3.13}$$

where c is a fixed constant. It follows that $Z_{0, \Lambda, \gamma}^{(N,r,s)}$ is the integral of a strictly positive function and hence > 0.

Thus (3.11) is well defined.

For $\xi \neq 0$, we have

$$\left| 1 + \frac{1}{2}\{\exp[\gamma(\rho_{y_k}^+, \phi) + i\phi(\rho_{y_k}^+)] + \exp[\bar{\gamma}(\rho_{y_k}^-, \phi) - i\phi(\rho_{y_k}^-)]\} \right| \leqslant 1 + \frac{c}{L_k^s} \tag{3.14}$$

From (3.11), (3.13), (3.14), and (3.3b) we get (see ref. 4, p. 161 for details)

$$\begin{aligned}
|G_{\xi,\Lambda}(x)| &\leqslant \sum_{\gamma\in\mathscr{F}} d_\gamma \frac{|Z^{(N,r,s)}_{\xi,\Lambda,\gamma}(x)|}{Z^{(N,r,s)}_{0,\Lambda,\gamma}} \\
&\leqslant \prod_{k=1}^{N}\left(\frac{1+cL_k^{-s}}{1-cL_k^{-s}}\right)^2 |W_0^{(N)}| \\
&\leqslant \left[\prod_{k=1}^{\infty}\left(\frac{1+cL_k^{-s}}{1-cL_k^{-s}}\right)^2\right]\prod_{k=1}^{N_0}\left(\frac{L_k}{15L_{k-1}}\right)^{-\beta\delta\eta^2/5\pi} \\
&\leqslant C'\left(\frac{L_{N_0}}{15^{N_0-1}L_1}\right)^{-\beta\delta\eta^2/5\pi} \\
&\leqslant C|x|_\infty^{-\theta\eta^2}
\end{aligned}$$

for some $\theta=\theta(\alpha, L_1, \beta)>0$.

This completes the proof of Theorem 1.1.

4. THE PROOF OF LEMMA 2.3

Let us fix $k\in\{1, 2,..., N-1\}$, $u\in\Lambda^{(k+1)}$, and recall $\tilde{\mathcal{N}}_{(k+1,u,r)}$ given by (2.12).

For $\rho\in\tilde{\mathcal{N}}_{(k+1,u,r)}$, let

$$\Gamma(\rho,\phi)=\log\{1+e^{\omega(\rho,\phi)}\cos[\phi(\rho)+\vartheta(\rho,\phi)]\} \tag{4.1}$$

Using the Taylor series for $\log(1+x)$ at $x>0$, plus $\cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta})$, we can write

$$\Gamma(\rho,\phi)=\sum_{m=1}^{\infty}\sum_{n\in J_m} a_{nm}e^{\gamma_{nm}(\rho,\phi)} \tag{4.2}$$

where $J_m=\{-m, -m+2, -m+4,..., m-2, m\}$,

$$a_{nm}=\frac{(-1)^{m+1}}{m\cdot 2^m}\,\frac{m!}{[(m-n)/2]!\,[(m+n)/2]!}$$

$$\gamma_{nm}(\rho,\phi)=m\omega(\rho,\phi)+in(\phi(\rho)+\vartheta(\rho,\phi))$$

and notice that

$$\sum_{n\in J_m}|a_{nm}|\leqslant\frac{1}{m}$$

and

$$|e^{\gamma_{nm}(\rho,\phi)}|=e^{m\omega(\rho,\phi)}$$

It thus follows from (2.7) that (4.2) is absolutely convergent if L_1 was sufficiently large.

Lemma 4.1. Let ρ be an (l, y, r)-admissible neutral charge density. Then $\Gamma(\rho, \phi)$ is a real-valued real-analytic function of the real variables $\{\phi(y');\ y' \in \bar{B}_l(y)\}$, such that for all $N = 1, 2,\dots$ and $\delta_1,\dots, \delta_N \in \mathscr{F}_l$ we have

$$\left| \prod_{j=1}^{N} \left(\delta_j \cdot \frac{\partial}{\partial \phi} \right) \Gamma(\rho, \phi) \right| \leqslant C_1 C^N N!\ |\rho|^N \prod_{j=1}^{N} d(\rho, \delta_j) \tag{4.3}$$

for some fixed constants $C < \infty$ and $C_1 = C_1(L_1, \alpha, r) < \infty$ such that $\lim_{L_1 \to \infty} C_1 = 0$.

Proof. Let $\mathscr{P}_N$ be the collection of partitions $\mathbf{P} = (P_1,\dots, P_s)$ of $\{1,\dots, N\}$; given a function f and derivations $D_1,\dots, D_N$, we have

$$D_{\{1,\dots,N\}} e^f = \sum_{\mathbf{P} \in \mathscr{P}_N} \left[\prod_{i=1}^{s} (D_{P_i} f) \right] e^f \tag{4.4}$$

where $D_Q = \prod_{j \in Q} D_j$.

Now let constants $C, C_1,\dots, C_N$ be given, and suppose for $Q \subset \{1, 2,\dots, N\}$ that we have

$$|D_Q f| \leqslant C C_Q r! \tag{4.5}$$

where $r = |Q|$ and $C_Q = \prod_{j \in Q} C_j$.

Then, if given $\mathbf{P} = (P_1,\dots, P_s)$ we let $r_i = |P_i|$, it follows from (4.4) and (4.5) that

$$|D_{\{1,\dots,N\}} e^f| \leqslant \left(\sum_{\mathbf{P} \in \mathscr{P}_N} C^s r_1! \cdots r_s! \right) |e^f|\, C_{\{1,\dots,N\}} \tag{4.6}$$

Let $T_N = \sum_{\mathbf{P} \in \mathscr{P}_N} C^s r_1! \cdots r_s!$. If $n_j = \#\{P_i : r_i = j,\ i = 1,\dots, s\}$, we have

$$T_N = N! \sum_{n_1,\dots,n_N} \frac{C^{n_1 + \cdots + n_N}}{n_1! \cdots n_N!}$$

where the summation is over all nonnegative integers $n_1,\dots, n_N$, which are solutions of the equation $n_1 + 2n_2 + \cdots + N n_N = N$. By using this constraint, T_N can be rewritten as

$$\begin{aligned} T_N &= 2^N N! \sum_{n_1,\dots,n_N} \frac{1}{n_1!} \left(\frac{C}{2} \right)^{n_1} \frac{1}{n_2!} \left(\frac{C}{2^2} \right)^{n_2} \cdots \frac{1}{n_N!} \left(\frac{C}{2^N} \right)^{n_N} \\ &\leqslant 2^N N! \exp\left(C \sum_{j=1}^{N} 2^{-j} \right) \\ &\leqslant 2^N N! \exp C \end{aligned}$$

Thus, it follows from (4.6) that

$$|D_{\{1,\dots,N\}} e^f| \leqslant 2^N N!\, e^C |e^f|\, C_{\{1,\dots,N\}} \tag{4.7}$$

We now use (4.2) and apply (4.4)–(4.7) to $f = \gamma_{nm}$, $D_j = \delta_j \cdot \partial/\partial\phi$, $j = 1, 2,\dots, N$. To do so, we must establish (4.5). We have

$$\left| \prod_{j \in Q} \left(\delta_j \cdot \frac{\partial}{\partial \phi} \right) \gamma_{nm} \right| \leqslant m \left| \prod_{j \in Q} \left(\delta_j \cdot \frac{\partial}{\partial \phi} \right) [\gamma(\rho, \phi) + i\phi(\rho)] \right|$$

$$\leqslant m \left[r!\, |\rho| \prod_{j \in Q} d(\rho, \delta_j) + \delta_{r,1} |\delta_{j_1}(\rho)| \right]$$

where we used $|n| \leqslant m$ and (2.5). Here $r = |Q|$, $\delta_{r,1} = 1$ if $r = 1$, so $Q = \{j_1\}$, and zero otherwise.

If $\delta = \varepsilon_v^{(l')} \in \mathscr{F}_l$, we use the neutrality of ρ to get

$$\delta(\rho) = \sum_x \delta(x)\, \rho(x) = \sum_x \rho(x)[\delta(x) - \delta(v)]$$

so

$$|\delta(\rho)| \leqslant |\rho| \sum_{x \in \operatorname{supp} \rho} |\delta(x) - \delta(v)| \leqslant c\, |\rho|\, d(\rho, \delta)$$

for some fixed constant $c < \infty$.

Thus, recalling that $|\rho| \geqslant 1$,

$$\left| \prod_{j \in Q} \left(\delta_j \cdot \frac{\partial}{\partial \phi} \right) \gamma_{nm} \right| \leqslant m \cdot r!\, |\rho|^r (1 + c)^r \prod_{j \in Q} d(\rho, \delta_j)$$

Thus, we can use (4.7) to conclude

$$\left| \prod_{j=1}^{N} \left(\delta_j \cdot \frac{\partial}{\partial \phi} \right) e^{\gamma_{nm}} \right| \leqslant N!\, |\rho|^N C^N \prod_{j=1}^{N} d(\rho, \delta_j) |e^{\omega + 1}|^m$$

where $C = 2(1 + c)$.

It now follows from (4.2) that

$$\left| \prod_{j=1}^{N} \left(\delta_j \cdot \frac{\partial}{\partial \phi} \right) \Gamma(\rho; \phi) \right| \leqslant N!\, |\rho|^N C^N \prod_{j=1}^{N} d(\rho, \delta_j) \sum_{m=1}^{\infty} \frac{1}{m} |e^{\omega + 1}|^m$$

which establishes (4.3).

Lemma 4.2. Let ρ be an (l, y, r)-admissible neutral charge density, $\delta \in \mathscr{F}_l$, κ real. Let $|\kappa|\, C\, |\rho|\, d(\rho, \delta) \leqslant 1/2$. We have

$$|\Re\{\Gamma(\rho, \phi + i\kappa\delta) - \Gamma(\rho, \phi)\}| \leqslant C_2 \kappa^2 |\rho|^2\, d(\rho, \delta)^2 \tag{4.8}$$

with $C_2 = C_2(L_1, \alpha, r) \to 0$ as $L_1 \to \infty$.

Proof. We have

$$|\Re\{\Gamma(\rho, \phi + i\kappa\delta) - \Gamma(\rho, \phi)\}| \leqslant \sum_{\substack{M=2 \\ M \text{ even}}}^{\infty} \frac{\kappa^M}{M!} \left| \left(\delta \cdot \frac{\partial}{\partial \phi}\right)^M \Gamma(\rho, \phi) \right|$$
$$\leqslant \sum_{\substack{M=2 \\ M \text{ even}}}^{\infty} C_1 [\kappa C |\rho| \, d(\rho, \delta)]^M$$
$$\leqslant C_2 \kappa^2 |\rho|^2 \, d(\rho, \delta)^2$$

provided $|\kappa| C |\rho| \, d(\rho, \delta) < 1/\sqrt{2}$, where $C_2 = 2C_1 C^2$, so $C_2 = C_2(L_1, \alpha, r) \to 0$ as $L_1 \to 0$.

Lemma 4.3. Under the assumptions of Lemmas 4.1 and 4.2, we have

$$\left| \prod_{j=1}^{N} \left(\delta_j \cdot \frac{\partial}{\partial \phi}\right) [\Gamma(\rho, \phi + i\kappa\delta) - \Gamma(\rho, \phi)] \right|$$
$$\leqslant |\kappa| \, 2C_1 (2C)^{N+1} (N+1)! \, |\rho|^{N+1} \prod_{j=1}^{N} d(\rho, \delta_j) \, d(\rho, \delta) \tag{4.9}$$

Proof. We have

$$\left| \prod_{j=1}^{N} \left(\delta_j \cdot \frac{\partial}{\partial \phi}\right) [\Gamma(\rho, \phi + i\kappa\delta) - \Gamma(\rho, \phi)] \right|$$
$$\leqslant \sum_{M=1}^{\infty} \frac{|\kappa|^M}{M!} \left| \prod_{j=1}^{N} \left(\delta_j \cdot \frac{\partial}{\partial \phi}\right) \left(\delta \cdot \frac{\partial}{\partial \phi}\right)^M \Gamma(\rho, \phi) \right|$$
$$\leqslant C_1 C^N |\rho|^N \prod_{j=1}^{N} d(\rho, \delta_j) \sum_{M=1}^{\infty} \frac{(M+N)!}{M!} [C |\kappa| \, |\rho| \, d(\rho, \delta)]^M \tag{4.10}$$

For $0 \leqslant \chi < 1/2$ we have

$$\sum_{M=1}^{\infty} \frac{(M+N)!}{M!} \chi^M = \left(\frac{d}{d\chi}\right)^N \frac{\chi^{N+1}}{1-\chi}$$
$$= \sum_{m=0}^{N} \binom{N}{m} \left[\left(\frac{d}{d\chi}\right)^m \chi^{N+1}\right] \left[\left(\frac{d}{d\chi}\right)^{N-m} \frac{1}{1-\chi}\right]$$
$$= N! \sum_{m=0}^{N} \binom{N+1}{m} \left(\frac{\chi}{1-\chi}\right)^{N+1-m}$$
$$= N! \left[\left(1 + \frac{\chi}{1-\chi}\right)^{N+1} - 1\right]$$
$$= N! \left[\frac{1}{(1-\chi)^{N+1}} - 1\right] \leqslant (N+1)! \, 2^{N+2} \chi \tag{4.11}$$

Using (4.11) in (4.10), we get (4.9).

Lemma 4.4. Let $y_0 \in \Lambda^{(k)}$, $\delta = \varepsilon_{y_0}^{(k)}$, $\tilde{B}_{k+1}(y_0) = B(y_0, \frac{1}{3}(L_{k+1} + 4L_k))$, $\tilde{B}_{k+1}^{(l)}(y_0) = \tilde{B}_{k+1}(y_0) \cap \Lambda^{(l)}$,

$$\hat{\mathcal{N}}_{(k+1, y_0, r)} = \bigcup_{y \in \tilde{B}_{k+1}^{(l)}(y_0)} \mathcal{N}_{(k, y, r)}.$$

Suppose κ is real and such that

$$|\kappa| \, C(\log L_{k-1})^p \frac{L_{k-1}}{(15/6) L_k - L_{k-1}} < \frac{1}{2} \tag{4.12}$$

Then

$$\left| \Re \left\{ \sum_{\rho \in \hat{\mathcal{N}}_{(k+1, y_0, r)}} [\Gamma(\rho, \phi + i\kappa\delta) - \Gamma(\rho, \phi)] \right\} \right| \leqslant \kappa^2 \pi b \log \frac{L_{k+1}}{15 L_k} \tag{4.13}$$

for some $b = b(\alpha, L_1, r)$ with $\lim_{L_1 \to 0} b = 0$.

Proof. Let $\hat{\mathcal{N}} = \hat{\mathcal{N}}_{(k+1, y, r)}$. By the definition of sparse neutral ensemble we can decompose $\hat{\mathcal{N}}$ according to the scale of its components, i.e., $\hat{\mathcal{N}}$ can be written as the disjoint union

$$\hat{\mathcal{N}} = \bigcup_{l=2}^{k} \hat{\mathcal{N}}^{(l)}$$

where

$$\hat{\mathcal{N}}^{(l)} = \{\rho_x^{(l)}; x \in \tilde{B}_{k+1}^{(l)}(y_0)\}$$

$\rho_x^{(l)}$ being an $(l-1, x', r)$-admissible neutral charge density for some $x' \in B_l^{(l-1)}(x)$, satisfying (2.7) and (2.8).

In view of (4.12) we can use Lemma 4.2 for each $\rho \in \hat{\mathcal{N}}$, obtaining

$$\left| \Re \left\{ \sum_{\rho \in \hat{\mathcal{N}}} [\Gamma(\rho, \phi + i\kappa\delta) - \Gamma(\rho, \phi)] \right\} \right| \leqslant C_2 \kappa^2 \sum_{\rho \in \hat{\mathcal{N}}^{(l)}} |\rho|^2 \, d(\rho, \delta)^2 \tag{4.14}$$

Now

$$\begin{aligned}
\sum_{\rho \in \hat{\mathcal{N}}} |\rho|^2 \, d(\rho, \delta)^2 &= \sum_{l=2}^{k} \sum_{\rho \in \hat{\mathcal{N}}^{(l)}} |\rho|^2 \, d(\rho, \delta)^2 \\
&\leqslant \sum_{l=2}^{k} \sum_{\substack{x \in \tilde{B}_{k+1}^{(l)}(y_0) \\ |x - y_0|_2 \geqslant (15/6) L_k}} (\log L_{l-1})^{2p} \left(\frac{L_{l-1}}{|x' - y_0|_2} \right)^2 \\
&\leqslant 4 \sum_{l=2}^{k} (\log L_{l-1})^{2p} \frac{L_{l-1}^2}{L_l^2} \sum_{t = 15 L_k / 6 L_l}^{L_{k+1}/6 L_l} \frac{4(2t+1)}{t^2} \\
&\leqslant C_3 \left[\sum_{l=2}^{k} (\log L_{l-1})^{2p} \frac{L_{l-1}^2}{L_l^2} \right] \log \left(\frac{1}{15} \frac{L_{k+1}}{L_k} \right) \\
&\leqslant C_4 \log \left(\frac{1}{15} \frac{L_{k+1}}{L_k} \right)
\end{aligned} \tag{4.15}$$

where C_3 is some fixed independent constant and $C_4 = C_4(\alpha, L_1) \to 0$ as $L_1 \to \infty$.

The result (4.13) now follows from (4.14) and (4.15).

Lemma 4.5. Let y_0, δ, κ, $\hat{\mathcal{N}} = \hat{\mathcal{N}}_{(k+1, y_0, r)}$ be as in Lemma 4.4. Let $\delta_j = \varepsilon_{x_j}^{(l_j)} \in \mathcal{F}_k$ for $j = 1, \dots, N$, where $N \in \{1, 2, \dots\}$. Then

$$\sum_{\rho \in \hat{\mathcal{N}}} \left| \prod_{j=1}^{N} \left(\delta_j \cdot \frac{\partial}{\partial \phi} \right) [\Gamma(\rho, \phi + i\kappa\delta) - \Gamma(\rho, \phi)] \right| \leqslant |\kappa| \tilde{C} N! \prod_{i=1}^{N} d(y_0, k+1, \delta_j) \tag{4.16}$$

where $\tilde{C} = \tilde{C}(L_1, \alpha, r, p) \to 0$ as $L_1 \to \infty$, and

$$d(y_0, k+1, \varepsilon_x^{(l)}) = \begin{cases} \dfrac{L_{k+1}}{|x - y_0|_2} & \text{if } \tilde{B}_{k+1}(y_0) \cap \left\{ u; \dfrac{15 L_l}{6} \leqslant |u - x|_2 \leqslant \dfrac{L_{l+1}}{6} \right\} \neq \varnothing \\ 0 & \text{otherwise} \end{cases} \tag{4.17}$$

Proof. By Lemma 4.3, the left-hand-side of (4.16) is

$$\leqslant |\kappa| 2C_1 (2C)^{N+1} (N+1)! \sum_{\rho \in \hat{\mathcal{N}}} |\rho|^{N+1} \prod_{j=1}^{N} d(\rho, \delta_j)\, d(\rho, \delta) \tag{4.18}$$

We have

$$\sum_{\rho \in \hat{\mathcal{N}}} |\rho|^{N+1} \prod_{j=1}^{N} d(\rho, \delta_j)\, d(\rho, \delta) = \sum_{l=2}^{k} \sum_{\rho \in \hat{\mathcal{N}}^{(l)}} |\rho|^{N+1} \prod_{j=1}^{N} d(\rho, \delta_j)\, d(\rho, \delta)$$

Let $\rho \in \hat{\mathcal{N}}^{(l)}$; we have $d(\rho, \delta_j) = 0$ unless (4.17) is satisfied. In that case, let ρ be localized in $B_{l-1}(x')$, $x' \in B_l^{(l-1)}(x)$, $x \in \tilde{B}_{k+1}^{(l)}(y_0)$. We have $d(\rho, \delta_j) = 0$ unless $|x - x_j|_2 \geqslant (15/6) L_{k+1} - L_k$. Thus

$$\tfrac{1}{2} |x - y_0|_\infty \leqslant \tfrac{1}{2} |x_j - y_0|_2 \leqslant |x_j - x'|_2$$

If $d(\rho, \delta) \neq 0$, we have $\frac{1}{2} |x - y_0|_2 \leqslant |x' - y_0|_2$. Thus, (4.18) is bounded by

$$\leqslant |\kappa| 2C_1 (2C)^{N+1} (N+1)! \times \sum_{\substack{x \in \tilde{B}_{k+1}^{(l)}(y_0) \\ |x - y_0|_2 \geqslant (15/6) L_k}} (\log L_{l-1})^{p(N+1)} L_{l-1}^{N+1} \frac{2}{|x - y_0|_\infty} \prod_{j=1}^{N} \frac{2}{|x_j - y_0|_2}$$

$$\leqslant |\kappa| 2C_1(N+1)! \prod_{j=1}^{N} \frac{1}{|x_j - y_0|_2}$$

$$\times \sum_{l=2}^{k} \frac{[4C(\log L_{l-1})^p L_{l-1}]^{N+1}}{L_l} \sum_{t=15L_k/6L_l}^{L_{k+1}/6L_l} \frac{4(2t+1)}{t}$$

$$\leqslant |\kappa| C_1 C_4 N! \prod_{j=1}^{N} \frac{L_{k+1}}{|x_j - y_0|_2} \left[(N+1) \sum_{l=2}^{k} \frac{[4C(\log L_{l-1})^p L_{l-1}]^{N+1}}{L_l^2 L_{k+1}^{N-1}} \right]$$

where C_4 is some independent constant. The term in brackets can be bounded by some constant $C_5(L_1, \alpha, r, p)$, such that $C_5 \to 0$ as $L_1 \to \infty$. Thus, (4.16) follows and the lemma is proved.

We are now ready to prove Lemma 2.3. Let $u_0 \in \Lambda^{(k+1)}$ with $\rho^*_{u_0} = \rho_{y_0}$ for some $y_0 \in B^{(k)}_{k+1}(u_0)$ and assume (2.18) holds, and let $\kappa > 0$. We obtain (2.19a) by performing the imaginary shift $\phi \to \phi + i\tilde{\kappa}\varepsilon^{(k)}_{y_0}$, with $\tilde{\kappa} = (q/|q|)\kappa$, and taking (see also the proof of Lemma 3.3 in ref. 4)

$$\bar{\rho}_{u_0} = \rho^*_{u_0} + \frac{\tilde{\kappa}}{\beta} \Delta\varepsilon^{(k)}_{y_0} \tag{4.19}$$

$$\gamma(\bar{\rho}_{u_0}, \phi) = \gamma(\rho^*_{u_0}, \phi) + \log \left\{ \frac{F(\hat{\mathcal{N}}_{(k+1, y_0, r)}; \phi + i\tilde{\kappa}\varepsilon^{(k)}_{y_0})}{F(\hat{\mathcal{N}}_{(k+1, y_0, r)}; \phi)} \right\} \tag{4.20}$$

$$Y^{(k)} = \left(\frac{L_{k+1}}{15L_k} \right)^{-\kappa|q|} \exp\left[\frac{\kappa^2}{2\beta} (\varepsilon^{(k)}_{y_0}, -\Delta\varepsilon^{(k)}_{y_0}) \right] \tag{4.21}$$

where $q = Q(\rho^*_{u_0}) \equiv \sum_y \rho^*_{u_0}(y)$, $\hat{\mathcal{N}}_{(k+1, y_0, r)}$ is defined in Lemma 4.4, and

$$F(\hat{\mathcal{N}}_{(k+1, y_0, r)}; \phi) = \prod_{\rho \in \hat{\mathcal{N}}} \{1 + e^{\omega(\rho,\phi)} \cos[\phi(\rho) + \vartheta(\rho, \phi)]\}$$

In (4.20) we used the fact that $\varepsilon^{(k)}_{y_0}$ is constant over $\bar{B}_k(y_0)$ and (2.5) to conclude that

$$\gamma(\rho^*_{u_0}, \phi + i\kappa\varepsilon^{(k)}_{y_0}) = \gamma(\rho^*_{u_0}, \phi) \tag{4.22}$$

Now (2.19b) follows from

$$0 \leqslant (\varepsilon^{(k)}_{y_0}, -\Delta\varepsilon^{(k)}_{y_0}) \leqslant \sum_{\substack{x, y \in \mathbf{Z}^2 \\ |x-y|_2 = 1}} [\varepsilon^{(k)}_{y_0}(x) - \varepsilon^{(k)}_{y_0}(y)]^2$$

$$\leqslant 2\pi \log \frac{L_{k+1}}{15L_k} + O(L_k^{-1})$$

$$\leqslant 2\pi \log \frac{L_{k+1}}{15L_k} (1 + a)$$

where $a = a(\alpha, L_1) \to 0$ as $L_1 \to \infty$.

Notice also that $\rho^*_{u_0}$ and $\Delta\varepsilon^{(k)}_{y_0}$ have disjoint supports, and

$$c'' \log L_{k+1} \leqslant |\Delta\varepsilon^{(k)}_{y_0}| \leqslant c' \log L_{k+1} \tag{4.23}$$

for some constants $0 < c'' \leqslant c' < \infty$.

Relation (2.20) follows from Lemma 4.4; (2.21) follows from Lemma 4.5. Notice that by (2.5), (4.16), and (4.20) we have

$$\left| \prod_{j=1}^{n} \left(\delta_j \cdot \frac{\partial}{\partial\phi} \right) \gamma(\bar{\rho}_{u_0}, \phi) \right| \leqslant n! \left[|\rho^*_{u_0}| \prod_{j=1}^{n} d(\rho^*_{y_0}, \delta_j) + \kappa\tilde{C} \prod_{j=1}^{n} d(y_0, k+1, \delta_j) \right]$$

This completes the proof of Lemma 2.3.

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